Miyazawa meet Christaller: Spatial multipliers within a Triple Decomposition of Input-output Central Place Systems

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Outline

1. Introduction
2. Structure of the Classical Central Place System
3. Application of Triangular Decomposition of the Leontief Inverse for IO-CP Systems
4. Decomposition of Leontief Inverse for IO-Top-Down CP Model
5. Decomposition of Leontief Inverse for IO-Bottom-Down CP Model
6. Miyazawa Income-Consumption Distribution in the IO Model
7. Miyazawa Income-Consumption Distribution in the IO-CP Model
8. Fine Structure of this System
9. Conclusions

1. Introduction: The Problem Posed

- Exploration of the temporal and spatial structure of input-output multipliers intrigued analysts for many decades
- In the time domain, recent initiatives focused on
  1. Temporal Leontief Inverse; sequential interindustry model
  2. Continuous time versions of econometric-input-output models from which the potential exists to extract hourly input-output tables….
- In the spatial domain, present paper offers perspective to link central place and input-output systems

Idea to investigate the input-output relationship within the central place system is not a new one:

- The necessity to combine together the hierarchical structure of central place system with the input-output structure of the transaction flows within one unifying framework was stressed in the programmatic paper of Isard (1960, p.141).
- Some ideas of central place theory useful for the description of the regional economic interaction was made by Chalmers *et al.*, 1978; however, no attention was directed to the intricate hierarchical structure of central place systems.
- A more systematic treatment of this problem was undertaken by Robison and Miller (1991). They used the rudimentary structure of (capital and neighboring cities) an intercommunity central place system, without paying attention to the fine structure of the central place hierarchy. The complexity of mathematical presentation was limited to the level of a simple two-community two-order sub region level with one dominant central place
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Present paper will attempt to:

- Integrate theory of central place hierarchies and multi-regional input-output analysis
- Reveal the way decompositions of the Leontief inverse for input-output central place systems reflect the process of complication of the evolving hierarchy of central places.
- Interpret classical Miyazawa interrelational multipliers as *spatial* multipliers in the developing structure of backward and forward linkages within three-level hierarchical central place systems of towns, cities and central capital.

2. Structure of the Classical Central Place System.

- The spatial description of the original Christaller central place model is based on three generic geometric properties of central places associated this central place system (see Sonis, 1986):

  1. The first property is that all hinterland areas of the central places at the same hierarchical level form a hexagonal covering of the plane with the centers on the homogeneous triangular lattice;

  2. The second property is that the size of the hinterland areas increases from the smallest (on the lower tier of the central place hierarchy) to the largest (on the highest tier of hierarchy) by a constant nesting factor *k*. This nesting factor expresses one of the Christaller's three principles, namely, marketing (*k*=3), transportation (*k*=4) and administrative (*k*=7) principles;

  3. The third property is that the center of a hinterland area of a given size is also the center of an hinterland of each smaller size (Christaller, 1933, Sonis, 1985)

- The Beckmann-McPherson (1970) central place model differs from the Christaller framework by applying variable nesting factors and by using the Löschian principle of all possible coverings of the plane by hexagons of variable integer sizes. Their centers are the vertices of the initial Christaller triangular lattice (Lösch, 1940).
- In this paper a stylized example of the Beckmann-McPherson three tier Central place system will be used, including an urban hierarchy:
    1. single large central place K and the hierarchical levels including
    2. cities $C = \{C_1, C_2, C_3, C_4, C_5, C_6\}$ and
    3. towns $T = \{T_1, T_2, T_3, T_4, T_5, T_6\}$ (see figure 1).
- Finally, the system is assumed to be for a closed economy with just one complete hierarchy; hence, there is no external trade.

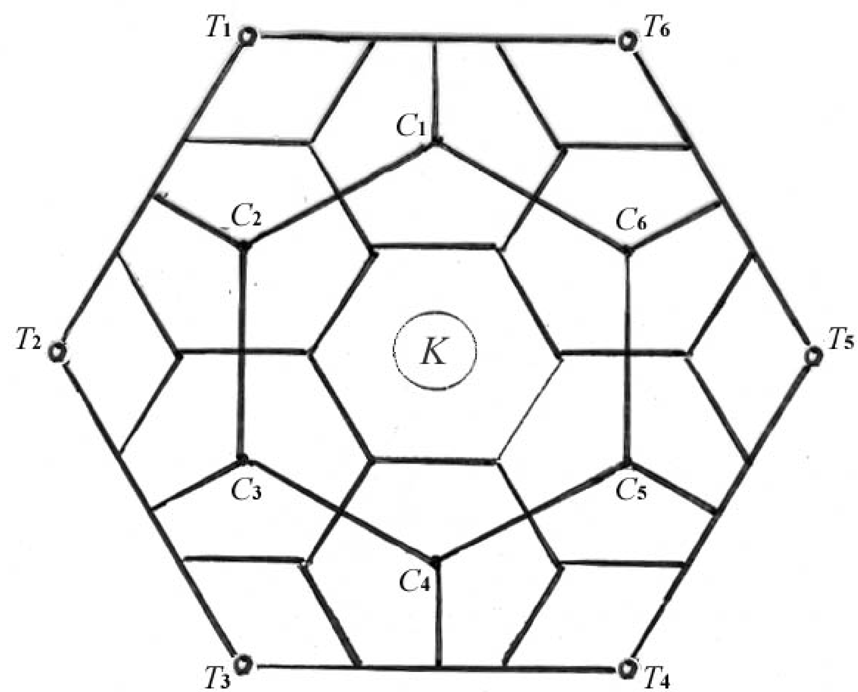


Figure 1. Beckmann-McPherson three tier Central Place system

3. The Application of the Triangular Decomposition of Leontief Inverse for the Analysis of the Input-Output Central Place System

- To begin, the central place spatial organization of settlements is set aside and only the hierarchy and the flow of intermediate goods between these three hierarchical levels will be considered.
- In such a case, the matrix of direct inputs can be presented in the form:

$$A=\begin{bmatrix} A_{TT} & A_{TC} & A_{TK} \\ A_{CT} & A_{CC} & A_{CK} \\ A_{KT} & A_{KC} & A_{KK} \end{bmatrix} \qquad (1)$$

For the analysis of such an input-output system, the following structure of the Leontief inverse $B=\left(I-A\right)^{-1}$ will be used together with the a triangular decomposition (*cf* Sonis and Hewings, 2000):

$$B=(I-A)^{-1}=\begin{bmatrix} I & & \\ B_{CC}^{2}(C,K)A_{CT}^{3} & I & \\ B_{KK}^{2}(C,K)A_{KT}^{3} & B_K A_{KC} & I \end{bmatrix}\begin{bmatrix} B_{TT}^{3} & & \\ & B_{CC}^{2}(C,K) & \\ & & B_K \end{bmatrix}\begin{bmatrix} I & A_{TC}^{3}B_{CC}^{2}(C,K) & A_{TK}^{3}B_{KK}^{2}(C,K) \\ & I & A_{CK}B_K \\ & & I \end{bmatrix} \quad (2)$$

In the formula (2) the following definitions of components are used:

1) The augmented inputs (see Yamada and Ihara, 1969) representing the coinfluence of different hierarchical levels (see Sonis and Hewings, 1998) have a form:

$$\begin{aligned} A_{TC}^{3} &= A_{TC} + A_{TK} B_K A_{KC} \\ A_{CT}^{3} &= A_{CT} + A_{CK} B_K A_{KT} \\ A_{KT}^{3} &= A_{KT} + A_{KC} B_C A_{CT} \\ A_{TK}^{3} &= A_{TK} + A_{TC} B_C A_{CK} \end{aligned} \quad (3)$$

2) The Leontief inverses of the economies on the three separate hierarchical levels of the Beckmann-McPherson Central place system are

$$B_T=(I-A_{TT});B_C=(I-A_{CC});B_K=(I-A_{KK}) \quad (4)$$

3) The extended Leontief inverses of the hierarchical level of intermediate cities C under the influence of the central city K and of central city K under influence of intermediate cities C are

$$B_{CC}^{2}\left(K,C\right)=\left[I-A_{CC}-A_{CK}B_{K}A_{KC}\right]^{-1};B_{KK}^{2}\left(K,C\right)=\left[I-A_{KK}-A_{KC}B_{C}A_{CK}\right]^{-1} \tag{5}$$

4) The component

$$B_{TT}^{3}=\left[I-A_{TT}-A_{TC}B_{CC}^{2}\left(K,C\right)A_{CT}^{3}-A_{TK}B_{KK}^{2}\left(K,C\right)A_{KT}^{3}\right]^{-1} \tag{6}$$

represents the extended Leontief inverse of the hierarchical level of towns, T, under the influence of the central city K and the intermediate cities C.

Further, the input-output partial central place system, including only two hierarchical levels of central city K and intermediate cities C and not including the level of towns T, corresponds to the block matrix of direct inputs:

$$A(K,C)=\begin{bmatrix} A_{CC} & A_{CK} \\ A_{KC} & A_{KK} \end{bmatrix} \qquad (7)$$

Its Leontief inverse has a form

$$B(K,C)=\left[I-A(K,C)\right]^{-1}=\begin{bmatrix} B_{CC}^{2}(K,C) & B_{CC}^{2}(K,C)A_{CK}B_{K} \\ B_{KK}^{2}(K,C)A_{KC}B_{C} & B_{KK}^{2}(K,C) \end{bmatrix}$$
(8)

Its the triangular decomposition (see Sonis and Hewings, 2000, p.571) is

$$B(K,C)=\begin{bmatrix} I & \\ B_{K}A_{KC} & I \end{bmatrix}\begin{bmatrix} B_{CC}^{2}(K,C) & \\ & B_{K} \end{bmatrix}\begin{bmatrix} I & A_{CK}B_{K} \\ & I \end{bmatrix}=$$
(9)
$$=\begin{bmatrix} I & \\ L_{KC} & I \end{bmatrix}\begin{bmatrix} B_{CC}^{2}(K,C) & \\ & B_{K} \end{bmatrix}\begin{bmatrix} I & U_{CK} \\ & I \end{bmatrix}$$

The backward and forward linkages L_{KC}, U_{CK} together generate the feedback loop of the form: $K \to C \to K$

In the triangular decomposition (2) the lower triangular matrix

$$L = \begin{bmatrix} I & & \\ B_{CC}^2(T)A_{CT}^3 & I & \\ B_{KK}^2(T)A_{KT}^3 & B_K A_{KC} & I \end{bmatrix} = \begin{bmatrix} I & & \\ L_{CT} & I & \\ L_{KT} & L_{KC} & I \end{bmatrix} \qquad (10)$$

represents the backward linkages of all three hierarchical levels such that the second block column of *L*:

$$\begin{bmatrix} 0 \\ I \\ B_K A_{KC} \end{bmatrix} = \begin{bmatrix} 0 \\ I \\ L_{KC} \end{bmatrix}$$

represents the new backward linkages of the subsystem of two hierarchical levels of central city *K* and intermediate cities *C* appearing as a result of adding to the economics of central city *K* the economics of intermediate cities *C*; the first column of the matrix *L:*

$$\begin{bmatrix} I \\ B_{CC}^{2}\left(T\right)A_{CT}^{3} \\ B_{KK}^{2}\left(T\right)A_{KT}^{3} \end{bmatrix} = \begin{bmatrix} I \\ L_{CT} \\ L_{KT} \end{bmatrix}$$

represents the extension of the *backward linkages* within the system *C* appearing as a result of further extension to the economies of central city *K* and intermediate cities *C* through the addition of an additional hierarchical - the towns *T*.

Analogously, the upper triangular matrix

$$U = \begin{bmatrix} I & A_{TC}^{3}B_{CC}^{2}\left(T\right) & A_{TK}^{3}B_{KK}^{2}\left(T\right) \\ & I & A_{CK}B_{K} \\ & & I \end{bmatrix} = \begin{bmatrix} I & U_{TC} & U_{TK} \\ & I & U_{CK} \\ & & I \end{bmatrix} \qquad (11)$$

represents the growing system of forward linkages within the augmenting system of all three hierarchical levels.

The backward and forward augmentation of linkages L_{CT}, U_{TC}; L_{KT}, U_{TK}; and L_{KC}, U_{CK} present three feedback loops of spatial economic dependencies:

$$C \rightarrow T \rightarrow C$$

$$K \rightarrow T \rightarrow K$$

$$K \rightarrow C \rightarrow K$$

The diagonal block matrix

$$D = \begin{bmatrix} B_{TT}^{3} & & \\ & B_{CC}^{2}(T) & \\ & & B_{K} \end{bmatrix} \qquad (12)$$

represents the Leontief inverse of the hierarchical level of the central city *K* and the extended Leontief inverses corresponding to the augmentation of the level *K* for the two hierarchical levels *K*, *C* and for the three hierarchical levels *K*, *C*, *T*.

The augmentation of the backward and forward linkages in three-tier hierarchical system presents the stages of development of the Central Place system starting from:

- one central place (capital) and
- gradually expending to two hierarchical levels (capital and neighboring cities) and then to
- three hierarchical levels (capital, neighboring cities and surrounding towns)

In the two following sections, the decomposition will be presented for a top-down and bottom up central place model respectively.

4. Decomposition of Leontief Inverse for Input-Output "Top-down" Central Place Model

- If the forward linkages in the system *K*, *C*, *T* are negligible (as implied by a strict central place system), then the matrix of direct inputs will have the form:

$$A = \begin{bmatrix} A_{TT} & 0 & 0 \\ A_{CT} & A_{CC} & 0 \\ A_{KT} & A_{KC} & A_{KK} \end{bmatrix} \qquad (13)$$

and the triangular representation (2) will include only backward linkages:

$$B = \begin{bmatrix} I & & \\ B_C A_{CT} & I & \\ B_K(A_{KT} + A_{KC} B_C A_{CT}) & B_K A_{KC} & I \end{bmatrix} \begin{bmatrix} B_T & & \\ & B_C & \\ & & B_K \end{bmatrix} \qquad (14)$$

This decomposition corresponds to:

- the circulation of flows of intermediate flows within the central places and
- “top-down” transaction flows where each central place includes the same structure of industries and sends “transaction flows” of direct inputs to:
    - each dependent central place within its own hinterland area and (e.g. cities to towns)
    - exchanging production with the central places of the same hierarchical tier (e.g., towns to towns)

5. Decomposition of the Leontief Inverse for Input-Output "Bottom-up" Central Place Model

The "bottom-up" system of transaction flows corresponds to the case of negligible backward linkages with the following block-matrix of direct inputs:

$$A = \begin{bmatrix} A_{TT} & A_{TC} & A_{TK} \\ 0 & A_{CC} & A_{CK} \\ 0 & 0 & A_{KK} \end{bmatrix} \qquad (15)$$

The corresponding triangular decomposition of the Leontief inverse will include only forward linkages:

$$B = \begin{bmatrix} B_T & & \\ & B_C & \\ & & B_K \end{bmatrix} \begin{bmatrix} I & (A_{TC} + A_{TK} B_K A_{KC}) B_C & A_{TK} B_K \\ & I & A_{CK} B_K \\ & & I \end{bmatrix} \qquad (16)$$

- The block-structure form of the triangular decomposition (2) allows the incorporation of the fine structure of the spatial economic dependencies between central places within different hierarchical levels.
- Each such specifications of this fine structure can be incorporated into the triangular decomposition (2).
- Next, consideration will be given to the income generated from production and the impact of its expenditures on goods and services produced in the system.
- While exchange in production need not be hierarchical, it will be assumed that consumption of goods is made according to the usual central place principles of seeking the good at the nearest place in the hierarchy that it is offered.
- In a more real world situation in which residents may not work and live in the same location, a more complex pattern of circulation can arise (see Hewings *et al*. 2003)

6. Miyazawa Income-Consumption Distribution in the Input-Output Model

Miyazawa (1968, and 1976 for the most complete exposition) introduced the matrix model:

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} A & C \\ V & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} f \\ g \end{bmatrix} \qquad (17)$$

with a block matrix $M = \begin{bmatrix} A & C \\ V & 0 \end{bmatrix}$ for the analysis of the interrelationships among various income groups in the process of income formation.

In the Miyazawa interpretation, the matrix A represents the inter industry direct inputs; vector *x* is gross output, vector *f* is final demand excluding consumption expenditures), vector *Y* represents the total income, the matrix V represents the value-added ratios; and the matrix C represents the coefficients of consumption expenditures.

Miyazawa considered the usual Leontief interindustry inverse $B = (I - A)^{-1}$;

the matrix multiplier VB showing the induced income earned from production activities among industries;

the matrix multiplier BC showing the induced production due to endogenous consumption (per unit of income) in each household sector;

the matrix multiplier $L = VBC$ showing interrelationships among incomes through the process of propagation from consumption expenditures.

The matrix $K = \left(I - L\right)^{-1} = (I - VBC)^{-1}$ is interpreted as the *interrelational income multiplier*, and the matrix multiplier KVB is interpreted as the matrix multiplier of income formation.

Further, the following Frobenius-Schur form:

$$B_e = (I - A - CV)^{-1} \qquad (18)$$

may be referred to as the enlarged Leontief inverse; its properties are:

$$KVB = VB_e; \quad BCK = B_e C \qquad (19)$$

Given the Miyazawa income generation scheme based on block matrix $M = \begin{bmatrix} A & C \\ V & 0 \end{bmatrix}$ the Leontief block-inverse has the following decomposition that separates the *backward* and *forward linkages* effects:

$$B(M) = \left(I\text{-}M\right)^{-1} = \begin{bmatrix} B_e & B_e C \\ VB_e & K \end{bmatrix} = \begin{bmatrix} B_e & BCK \\ KVB & K \end{bmatrix} \qquad (20)$$

The structure of the interconnection between the interrelational income multiplier *K* and the enlarged Leontief inverse B_e can be seen and interpreted from the following formula:

$$K = I + VB_e C \qquad (21)$$

where the enlarged matrix multiplier $L_e = VB_e C$ shows the interrelationships among incomes through the process of enlarged propagation from consumption expenditures.

Next, the fundamental *UDL*-factorization

$$B(M) = \left(I\text{-}M\right)^{-1} = \begin{bmatrix} B_e & BCK \\ KVB & K \end{bmatrix} = \begin{bmatrix} I & BC \\ 0 & I \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} I & 0 \\ VB & I \end{bmatrix} \qquad (22)$$

separates multiplicatively:

- the induced income earned from production activities among industries, *VB*
- the induced production due to endogenous consumption (per unit of income) in each household sector *BC* from the industrial production activities.

In this factorization, the interrelational income multiplier, *K*, appears explicitly and the enlarged Leontief inverse, B_e, appears implicitly.

The structure of the interconnection between the enlarged Leontief inverse, B_e, and the interrelational income multiplier, *K*, can be seen and interpreted from the following formula:

$$B_e = B + BCKVB \qquad (23)$$

7. Miyazawa Income-Consumption Distribution in Input-Output Central Place Model

The Miyazawa income-consumption distribution in the input-output hierarchical central place model can be presented with the help of the following matrix:

$$M = \begin{bmatrix} A_{TT} & A_{TC} & A_{TK} & C_T \\ A_{CT} & A_{CC} & A_{CK} & C_C \\ A_{KT} & A_{KC} & A_{KK} & C_K \\ V_T & V_C & V_K & \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{C} \\ \tilde{V} & \end{bmatrix} \qquad (24)$$

where the matrix $\widetilde{A}$ is the block matrix of direct inputs on different hierarchical levels of towns, cities and capital city,

$\tilde{V} = \begin{pmatrix} V_T & V_C & V_K \end{pmatrix}$ represents the income corresponding to the different sectors in towns, cities and capital city and $\tilde{C} = \begin{pmatrix} C_T \\ C_C \\ C_K \end{pmatrix}$ represents the consumption in the central place hierarchy.

Analogously to formula (2), the following formula provides the following triple decomposition:

$$B(M) =$$

$$= \begin{bmatrix} I & & & \\ & I & & \\ & & I & \\ V_T & V_C & V_K & I \end{bmatrix} \begin{bmatrix} I & & & \\ L_{CT} & I & & \\ L_{KT} & L_{KC} & I & \\ & & & I \end{bmatrix} \begin{bmatrix} D_T & & & \\ & D_C & & \\ & & D_K & \\ & & & I \end{bmatrix} \begin{bmatrix} I & U_{TC} & U_{TK} & \\ & I & U_{CK} & \\ & & I & \\ & & & I \end{bmatrix} \begin{bmatrix} I & & & C_T \\ & I & & C_C \\ & & I & C_K \\ & & & I \end{bmatrix} =$$

$$= \begin{bmatrix} L & \\ \tilde{V}L & I \end{bmatrix} \begin{bmatrix} D & \\ & I \end{bmatrix} \begin{bmatrix} U & U\tilde{C} \\ & I \end{bmatrix} \qquad (25)$$

where induced income and induced production are

$$\tilde{V}L = \begin{pmatrix} V_T & V_C & V_K \end{pmatrix} \begin{bmatrix} I & & \\ L_{CT} & I & \\ L_{KT} & L_{KC} & I \end{bmatrix} = \begin{pmatrix} V_T + V_C L_{CT} + V_K L_{KT} & V_C + V_K L_{KT} & V_K \end{pmatrix}$$

$$U\tilde{C} = \begin{bmatrix} I & & \\ U_{CT} & I & \\ U_{KT} & U_{KC} & I \end{bmatrix} \begin{bmatrix} C_T \\ C_C \\ C_K \end{bmatrix} = \begin{bmatrix} C_T \\ U_{CT} C_T + C_C \\ U_{KT} C_T + U_{KC} C_C + C_K \end{bmatrix} \quad (26)$$

Therefore, the Miyazawa interrelation income multiplier (see 23) will have the form:

$$K = I + \begin{pmatrix} V_T & V_C & V_K \end{pmatrix} \begin{bmatrix} I & & \\ L_{CT} & I & \\ L_{KT} & L_{KC} & I \end{bmatrix} \begin{bmatrix} D_T & & \\ & D_C & \\ & & D_K \end{bmatrix} \begin{bmatrix} I & & \\ U_{CT} & I & \\ U_{KT} & U_{KC} & I \end{bmatrix} \begin{bmatrix} C_T \\ C_C \\ C_K \end{bmatrix} =$$

$$= \begin{pmatrix} V_T + V_C L_{CT} + V_K L_{KT} & V_C + V_K L_{KT} & V_K \end{pmatrix} \begin{bmatrix} D_T & & \\ & D_C & \\ & & D_K \end{bmatrix} \begin{bmatrix} C_T \\ U_{CT} C_T + C_C \\ U_{KT} C_T + U_{KC} C_C + C_K \end{bmatrix}$$

(27)

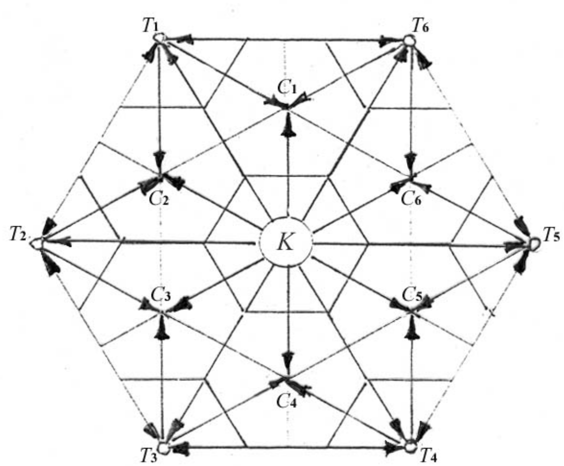
Analogously to the spatial feedback loop interpretation of the formula (2), the Miyazawa interrelation income multiplier can be interpreted as a *spatial multiplier in the central place input-output system*.

8. The Fine Structure of the Miyazawa Income-Consumption Distribution in Input-Output Central Place Model

Next, a simple case of the structure will be presented that includes:

- the circulation of flows of intermediate goods within the central places and
- "top-down" structure of the transaction flows between the dependent central places only.

Such a structure is presented in figure 2: *"Top-down" structure of transaction flows in input-output three tier Beckmann-McPherson central place system*



This scheme generalizes slightly the usual assumption of classical central place theory demanding that both income earned from production and consumption expenditures from this income will be concentrated in same location.

As noted earlier, real world considerations would suggest that there would be greater circulation

Even in a strict hierarchy, residents working in a different level in the hierarchy from their residence may distort the system by purchasing lower-order goods in a higher order place

The corresponding block-matrix of direct inputs reflecting the central place hierarchy of such system has a form:

$$A = \begin{bmatrix} A_{TT} & A_{TC} & A_{TK} \\ A_{CT} & A_{CC} & A_{CK} \\ A_{KT} & A_{KC} & A_{KK} \end{bmatrix} =$$

$$= \left[\begin{array}{ccccccccccccc}
a_{T_1T_1} & & & & & & & & & & & & \\
& a_{T_2T_2} & & & & & & & & & & & \\
& & a_{T_3T_3} & & & & & & & & & & \\
& & & a_{T_4T_4} & & & & & & & & & \\
& & & & a_{T_5T_5} & & & & & & & & \\
& & & & & a_{T_6T_6} & & & & & & & \\
a_{C_1T_1} & & & & & a_{C_1T_6} & a_{C_1C_1} & & & & & & \\
a_{C_2T_1} & a_{C_2T_2} & & & & & & a_{C_2C_2} & & & & & \\
& a_{C_3T_2} & a_{C_3T_3} & & & & & & a_{C_3C_3} & & & & \\
& & a_{C_4T_3} & a_{C_4T_4} & & & & & & a_{C_4C_4} & & & \\
& & & a_{C_5T_4} & a_{C_5T_5} & & & & & & a_{C_5C_5} & & \\
& & & & a_{C_6T_5} & a_{C_6T_6} & & & & & & a_{C_6C_6} & \\
a_{KT_1} & a_{KT_2} & a_{KT_3} & a_{KT_4} & a_{KT_5} & a_{KT_6} & a_{KC_1} & a_{KC_2} & a_{KC_3} & a_{KC_4} & a_{KC_5} & a_{KC_6} & A_{KK}
\end{array} \right] \quad (28)$$

The *LD*-substructure of the Leontief inverse for this input-output central place system has a form drawing on (14) that is as follows:

$$LD = \begin{bmatrix} I & & \\ B_C A_{CT} & I & \\ B_K (A_{KT} + A_{KC} B_C A_{CT}) & B_K A_{KC} & I \end{bmatrix} \begin{bmatrix} B_T & & \\ & B_C & \\ & & B_K \end{bmatrix}$$

where

$$B_T = \begin{bmatrix} b_{T_1T_1} & & & & & \\ & b_{T_2T_2} & & & & \\ & & b_{T_3T_3} & & & \\ & & & b_{T_4T_4} & & \\ & & & & b_{T_5T_5} & \\ & & & & & b_{T_6T_6} \end{bmatrix}, \qquad (29)$$

$$B_C = \begin{bmatrix} b_{C_1C_1} & & & & & \\ & b_{C_2C_2} & & & & \\ & & b_{C_3C_3} & & & \\ & & & b_{C_4C_4} & & \\ & & & & b_{C_5C_5} & \\ & & & & & b_{C_6C_6} \end{bmatrix} \quad (30)$$

where

$$b_{T_iT_i} = \left(I - a_{T_iT_i}\right)^{-1}; b_{C_iC_i} = \left(I - a_{C_iC_i}\right)^{-1}; i = 1, 2, ..., 6 \quad (31)$$

and

$$B_C A_{CT} = \begin{bmatrix} b_{C_1C_1}a_{C_1T_1} & & & & & b_{C_1C_1}a_{C_1T_1} \\ b_{C_1C_1}a_{C_1T_1} & b_{C_1C_1}a_{C_1T_1} & & & & \\ & b_{C_1C_1}a_{C_1T_1} & b_{C_1C_1}a_{C_1T_1} & & & \\ & & b_{C_1C_1}a_{C_1T_1} & b_{C_1C_1}a_{C_1T_1} & & \\ & & & b_{C_1C_1}a_{C_1T_1} & b_{C_1C_1}a_{C_1T_1} & \\ & & & & b_{C_1C_1}a_{C_1T_1} & b_{C_1C_1}a_{C_1T_1} \end{bmatrix}, \quad (32)$$

$$B_K A_{KT}^3 = B_K(A_{KT} + A_{KC}B_C A_{KC}) = \begin{pmatrix} b_{KT_1} & b_{KT_2} & b_{KT_3} & b_{KT_4} & b_{KT_5} & b_{KT_6} \end{pmatrix}$$

(33)

where $b_{KT_i} = B_K\left(a_{KT_i} + a_{KC_i}b_{C_iC_i}a_{C_iT_i} + a_{KC_{i+1}}b_{C_{i+1}C_{i+1}}a_{C_{i+1}T_i}\right), i = 1, 2, ..., 5,$

and $b_{KT_6} = B_K\left(a_{KT_6} + a_{KC_1}b_{C_1C_1}a_{C_1T_6} + a_{KC_6}b_{C_6C_6}a_{C_6T_6}\right)$

$$B_K A_{KC} = \begin{pmatrix} B_K a_{KC_1} & B_K a_{KC_2} & B_K a_{KC_3} & B_K a_{KC_4} & B_K a_{KC_5} & B_K a_{KC_6} \end{pmatrix} \quad (34)$$

So the places of non-zero blocks in the matrix L of the backward linkages corresponding to the scheme of backward linkages that was described in figure 2 is as follows:

$$\begin{bmatrix}
I & & & & & & & & & & & & \\
& I & & & & & & & & & & & \\
& & I & & & & & & & & & & \\
& & & I & & & & & & & & & \\
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\end{bmatrix} \quad (35)$$

Structure of income generation in this input-output central place system is given by following diagonal matrix reflecting the income generation within towns, cities and the central city:

$$\overline{V} = \begin{bmatrix} v_{T_1T_1} & & & & & & & & & & & & \\ & v_{T_2T_2} & & & & & & & & & & & \\ & & v_{T_3T_3} & & & & & & & & & & \\ & & & v_{T_4T_4} & & & & & & & & & \\ & & & & v_{T_5T_5} & & & & & & & & \\ & & & & & v_{T_6T_6} & & & & & & & \\ & & & & & & v_{C_1C_1} & & & & & & \\ & & & & & & & v_{C_2C_2} & & & & & \\ & & & & & & & & v_{C_3C_3} & & & & \\ & & & & & & & & & v_{C_4C_4} & & & \\ & & & & & & & & & & v_{C_5C_5} & & \\ & & & & & & & & & & & v_{C_6C_6} & \\ & & & & & & & & & & & & v_{KK} \end{bmatrix} \quad (36)$$

The consumption in this central place system is given by the following matrix whose block structure reflects the “bottom-up” spatial organization of consumption in which the income generated in the towns, cities and capital is spent also in the neighboring central places:

$$\widetilde{C} = \begin{bmatrix}
c_{T_1T_1} & c_{T_1T_2} & & & & c_{T_1T_6} & c_{T_1C_1} & & & & & c_{T_1C_6} & c_{T_1K} \\
c_{T_2T_1} & c_{T_2T_2} & c_{T_2T_3} & & & & c_{T_2C_1} & c_{T_2C_2} & & & & & c_{T_2K} \\
 & c_{T_3T_2} & c_{T_3T_3} & c_{T_3T_4} & & & & c_{T_3C_2} & c_{T_3C_3} & & & & c_{T_3K} \\
 & & c_{T_4T_3} & c_{T_4T_4} & c_{T_4T_5} & & & & c_{T_4C_3} & c_{T_4C_4} & & & c_{T_4K} \\
 & & & c_{T_5T_4} & c_{T_5T_5} & c_{T_5T_6} & & & & c_{T_5C_4} & c_{T_5C_5} & & c_{T_5K} \\
c_{T_6T_1} & & & & c_{T_6T_5} & c_{T_6T_6} & & & & & c_{T_6C_6} & c_{T_1T_1} & c_{T_6K} \\
c_{C_1T_1} & c_{C_1T_2} & & & & & c_{C_1C_1} & & & & & & c_{C_1K} \\
 & c_{C_2T_2} & c_{C_2T_3} & & & & & c_{C_2C_2} & & & & & c_{C_2K} \\
 & & c_{C_3T_3} & c_{C_3T_4} & & & & & c_{C_3C_3} & & & & c_{C_3K} \\
 & & & c_{C_4T_4} & c_{C_4T_5} & & & & & c_{C_4C_4} & & & c_{C_4K} \\
 & & & & c_{C_5T_5} & c_{C_5T_6} & & & & & c_{C_5C_5} & & c_{C_5K} \\
c_{C_6T_1} & & & & & c_{C_6T_6} & & & & & & c_{C_6C_6} & c_{C_6K} \\
c_{KT_1} & c_{KT_2} & c_{KT_3} & c_{KT_4} & c_{KT_5} & c_{KT_6} & c_{KC_1} & c_{KC_2} & c_{KC_3} & c_{KC_4} & c_{KC_5} & c_{KC_6} & c_{KK}
\end{bmatrix} \quad (37)$$

- The formulae (24-37) include all blocks of the Miyazawa interrelational income multiplier (27) and can be used for its analysis
- Provides capability to explore the structure of income propagation within central place systems
- Can be used to analyze the process of income distribution under various developmental conditions
  - Does the hierarchy sustain and perpetuate disparities?
  - Under what conditions would income convergence arise?
  - What are the critical parameters in conditioning these distributions?

9. Conclusions

- In this paper the attempt is described to unify two central theories in the regional science: the classical input-output theory of Leontief and the classical Christaller-Lösch central place theory.
- Further extensions to include the Miyazawa income formation-distribution-consumption relationships provide a way to explore properties of spatial distribution of income within an hierarchical system
- It is expected that the proposed theoretical methodology will be useful for the analysis of organization of the production economics in geographical space.
- Initial attempt to explore these properties in a stylized representation of the Chicago region economy